DISCRETE FACILITIES LOCATION AND ALLOCATION PROBLEM : A LAGRANGIAN RELAXATION APPROACH

by

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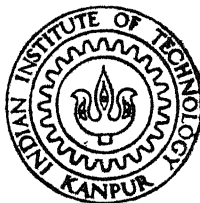
INDUSTRIAL AND MANAGEMENT ENGINEERING PROGRAMME

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BIMAL K. MODI

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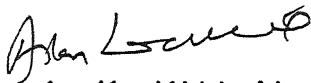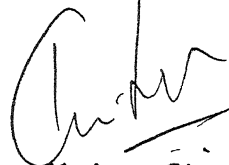
CERTIFICATE

13/2/91

It is certified that the work contained in the thesis entitled *"Discrete Facilities Location and Allocation Problem : A Lagrangian Relaxation Approach"*, by Bimal K. Modi (Roll No. : 8911402), has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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Bimal K. Modi

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ABSTRACT

The problem considered is location of base depots and allocation of sales depots to these base depots for each product with the objective of minimizing the total delivery cost to the customers' place such that the total demand is satisfied under the constrained capacity of base depots. The problem has been viewed in the perspective of the multiechelon distribution system of a consumer product company.

At the first stage the problem which is formulated as a 0-1 integer linear programme is solved using Lagrangian relaxation technique. After some fixed number of iterations of Lagrangian dual search, some values for location variables are obtained. These values are, then imposed as constraints to the original problem which in turn reduces the original problem to a generalized assignment problem. At the second stage this reduced problem which is GAP is solved using Balas's additive algorithm. At this stage, usually a good feasible solution to the original problem is obtained. Now with this solution as the upper bound on the optimum, branch and bound procedure is tried with Lagrangian relaxation as the lower bounding technique. The qualities of and the CPU-times required to obtain the first feasible solution and the optimum solution are compared. Some heuristic rules are also suggested for fast convergence of the Lagrangian dual search.

A procedure is also suggested for the overall optimization of the distribution system with this problem as a subproblem.

CHAPTER I

DISCRETE FACILITIES LOCATION AND ALLOCATION PROBLEM : A BRIEF INTRODUCTION

Facilities location and allocation problems have been the subject of analysis for decades. With the emergence of the operations research techniques, the subject is receiving wider attention. The area has been of interest to operations researchers, economists, urban planners, architects, engineers, sales managers, etc.. Each of them gives different interpretation to the problem with different terminologies, but the basic nature of the problem remains the same. In the present work, the problem is viewed from the point of view of a sales manager who wishes to optimize the distribution system.

The recent trend of increasing competitiveness in the market, demand for higher reliability of the delivery schedule, increasing delivery cost to the customers' place and multiple manufacturing sites with a wide range of products has compelled the sales managers to rethink the way in which they maintain their distribution system. They are facing several questions which need to be answered periodically. These questions can be divided into two broad categories depending upon their periodicity.

The first set of questions with long planning horizon (4-5 years) is:

(Q1) What should be the number of levels in the distribution system?

(Q2) Which products should be shipped from a plant?

(Q3)   What should be the route of shipment in the case of multiple echelons?

(Q4)   Where should the facilities belonging to the intermediate echelons be located?

The second set with short planning horizon (1-2 years) is:

(Q5)   What should be the customer service level at the lowest echelon of the distribution system?

(Q6)   What should be the service levels at the intermediate echelons for the next lower echelon?

(Q7)   How the inventory planning and control policies should be established?

The actual list is, in fact, very large. Here, only a subset is presented in order to have a feel of the imbroglio of a sales manager.

In the present work, an attempt has been made to answer the first set of questions such that the total delivery cost is minimized. For the sake of completeness, the other set of questions is also addressed and a way is suggested to answer these questions.

To make the things further clear in the successive chapters, the following terms are defined keeping in mind the distribution system (illustrated in figure 1.1) of a consumer product company.

Sales Depot

A sales depot is at the lowest echelon of the distribution system and the retailers are served from this place.

Base Depot

A base depot belongs to the intermediate echelons of the distribution system, i.e, between plants and sales depots. It receives the supplies from the plants and ships to the sales depots. When there are several intermediate echelons, the echelon number is also attached to the base depot, e.g, base depot-2 for base depot at the second echelon, base depot-3 for base depot at the third echelon for a four echelon distribution system.

The nature of the cost parameters, considered in this work, is in accordance with a consumer product company. The cost parameters are linear in the quantity of products handled. It may be pointed out that the carrying costs at the sales/base depots are on the basis of quantity stored, the duration of storage does not come into picture.

The present work addresses the discrete version of the facilities location allocation problem. Here the word discrete represents the conditions that a set of limited number of locations is available for potential base depots and the amount of a product required at a sales depot is shipped from the plant through a single route. The objective is to achieve the optimal locations of base depots from a possible set of potential locations and the product supply route from plant to sales depot such that the total cost is minimized simultaneously satisfying the following conditions:

(C1) The demand of all the products must be satisfied at all the sales depots so that these sales depots can satisfy their retailers' demands.

(C2) The amount handled at a base depot should be within its capacity limits. This condition can further bring the uniformity among base depots in terms of the quantity handled.

In the next chapter a brief review of the literature related to the above problem has been presented and in the subsequent chapters the problem formulation, the suggested solution methodologies, the implementation, related aspects, computational experience of the implementation and finally, conclusion are presented.

ECHELON 1 : Plants

Plant-1	Plant-2	Plant-3		Plant-L
Products	Products	Products	. . .	Products
↓	↓	↓		↓
P1,P2,P3	P1,P2,P4	P3,P4,P5		P3,P4,Pp

| | SUPPLY | | | SUPPLY | | | SUPPLY | | | SUPPLY | |
| ↓ ↓ | | | ↓ ↓ | | | ↓ ↓ | | | ↓ ↓ | |

ECHELON 2 : Base Depots

[1]   [2]   [3] . . . . [m]

| | SUPPLY | | | SUPPLY | | | SUPPLY | | | SUPPLY | |
| ↓ ↓ | | | ↓ ↓ | | | ↓ ↓ | | | ↓ ↓ | |

ECHELON 3 : Sales Depots

[1]   [2]   [3] . . . . [n]
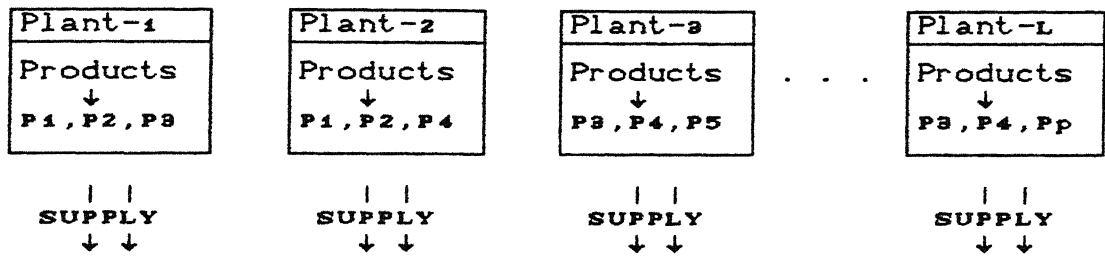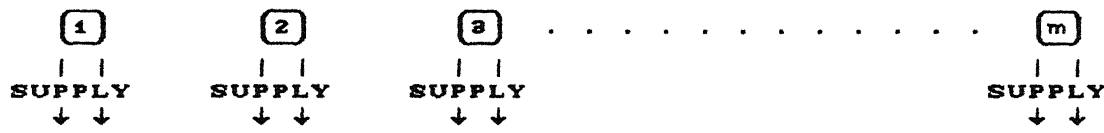
↑↑↑	↑↑↑	↑↑↑		↑↑↑
DEMAND	DEMAND	DEMAND		DEMAND
I I I	I I I	I I I		I I I
Retailers	Retailers	Retailers		Retailers

Figure 1.1 : A 3-Echelon Distribution System

CHAPTER II
LITERATURE SURVEY AND SCOPE OF THE PRESENT WORK

In this chapter, first the literature related to the facilities location and allocation problems is reviewed, and then in the next section the scope of the thesis is outlined briefly.

2.1 REVIEW OF LITERATURE

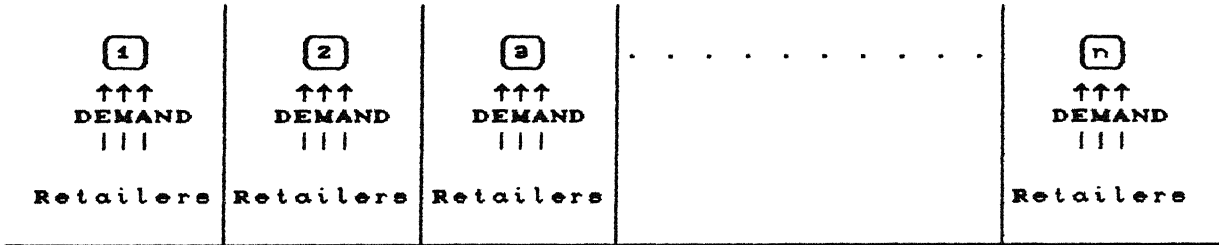The problem considered in this thesis broadly comes under the category of facilities location and allocation problems. A substantial amount of research has been done in this area and Sharma [16] provides an excellent source of references. The most general problem in this area is the capacitated facility location problem and the others are its variants. The capacitated facility location problem (CFL) is presented below.

$$(CFL) : \text{Minimize} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} X_{ij} + \sum_{i=1}^{m} F_i Y_i \qquad (2.1)$$

s.t.

$$\sum_{i=1}^{m} X_{ij} = 1 \qquad \qquad \forall j \qquad (2.2)$$

$$X_{ij} \leq Y_i \qquad \qquad \forall i, \forall j \qquad (2.3)$$

$$\sum_{j=1}^{n} D_j X_{ij} \leq S_i Y_i \qquad \forall i \qquad (2.4)$$

$$X_{ij} \geq 0, \quad Y_i \in \langle 0,1 \rangle \qquad \forall i, \forall j, \qquad (2.5)$$

where, m = the total number of possible locations

n = the total number of demand points

D_j = the demand of the customer j

F_i = the fixed cost of locating a facility at i

S_i = the limit on the supply from the facility i

C_{ij} = cost of supplying customer j's demand from facility i

$$Y_i = \begin{cases} 1, \text{ if a facility is located at } i, \\ 0, \text{ otherwise} \end{cases}$$

X_{ij} = the fraction of the customer j's demand delivered by facility i.

In the above formulation all demands must be met (constraint 2.2) and total supply from a facility i is limited by S_i (constraint 2.4). The above formulation of CFL is known as *strong formulation*. It is possible to obtain a more compact formulation by replacing the variable upper bound constraints (2.3) with

$$\sum_{j=1}^{n} X_{ij} \leq nY_i \qquad \forall \ i. \qquad (2.6)$$

The very first formulation of the uncapacitated version of this problem, i.e., by dropping the constraints (2.4) has been suggested by Efroymson and Ray [4]. The solution procedure employed by Efroymson and Ray is to first solve the problem as a linear programme, ignoring the integer restriction on Y_i. Obviously, if all the Y_i values obtained by solving the linear programme are integer valued, then the solution is attained. Otherwise, a branch and bound procedure is employed. It is to be noted that the branch and bound adopts the *weaker formulation* i.e., with constraints (2.6). A variety of node selection rules and branching rules suggested by Khumawala are discussed in Francis and White [7].

Erlenkotter [5] has developed and tested a method for the uncapacitated version of CFL that is based on a linear

programming dual formation. His dual ascent and adjustment procedure frequently produces dual optimal solution. If not, then he uses a branch and bound procedure. He reports excellent computational results.

Cornuejols, Fisher and Nemhauser [2] have also treated the uncapacitated version of CFL with X_{ij} restricted to the set {0,1}. In their exceptional paper, they have presented a Lagrangian relaxation by relaxing the constraint (2.2) which provides a lower bound on the solution. They have further suggested heuristics to obtain upper bounds on the optimal solution. Their main results are worst case analyses of these bounds.

Geoffrion [8] has suggested Lagrangian relaxation relative to constraints (2.2) for solving CFL . Afterwards in a later paper, Geoffrion and McBride [9] have given detailed methodology with computational results. The Lagrangian problem decomposes into m continuous knapsack problems.

Van Roy [17] has presented an implementation of the cross decomposition method to solve CFL. The method unifies Bender's decomposition and Lagrangian relaxation into a single framework. The methodology involves solutions to a transportation problem and an uncapacitated version of CFL. He has used the dual ascent method suggested by Erlenkotter for solving the uncapacitated version of CFL. He has also presented a simple algorithm to strengthen the Bender's cut.

There are several exact algorithms and heuristics available for CFL. Within this small framework it is not possible to discuss all of them. Sharma [16], in general, provides a good source of references related to CFL and its variants. He has also presented a brief review of most of these heuristics and algorithms. His work mainly deals with multistage bulk commodity distribution.

As discussed in Chapter I, the problem considered in this thesis is of slightly different nature.

First, the problem discussed has an increased number of dimensions in decision variables.

Second, the continuous variables of the formulation of CFL presented above are restricted to take the values 0 or 1 only, because of the following reason. A consumer product company, usually employs the mode of goods' transport as the surface transport and that also by trucks only. This type of transportation system is owned by private agencies. The charges are different for full truck load (FTL) and part truck load (PTL). For PTL, the charges are higher than that for FTL and there is no guarantee of transit time, but usually it takes additional 2-3 days. The damage to the goods is also much more in the case of PTL than that in FTL transport. Therefore, most of the consumer product companies have adopted the policy of avoiding the PTL.

Thus, by restricting the continuous variables of the formulation of CFL presented above to the set {0,1}, the above mentioned purpose is served but the problem becomes very hard to solve. The increased number of dimensions in decision variables also adds to the computational complexity of the problem.

2.2 SCOPE OF THE THESIS

The present thesis deals with discrete facilities location and allocation problems. A Lagrangian relaxation based branch and bound procedure has been suggested for the solution of the discrete location and allocation problem with the objective of minimizing the total delivery cost incurred while shipping the products from plant to the customers' place. With the computational complexity of the problem in view, the thesis also aims at suggesting the ways to obtain good heuristic solutions before starting the branch and bound procedure.

The branch and bound procedure is very much special to this problem, because only a few variables which denote the location of a base depot are selected as branching variables. The complete formulation and solution methodology are discussed in Chapter III.

As pointed out in Chapter I, a sales manager faces not only the questions with long planning horizon but also the questions with short planning horizon. This thesis mainly deals with the former category of questions. Nevertheless, in Chapter IV, the related aspects of the problem which are in the later category

are discussed. In the same chapter, some of the experiences based on computational experience are listed. Some of these observed characteristics are analyzed and explained.

Conclusions that could be drawn from the developed formulations and procedures, and some suggestions for future research have been presented in Chapter V.

CHAPTER III
PROBLEM FORMULATION AND SOLUTION METHODOLOGY

In this chapter, the formulation of the discrete facilities location and allocation problem is presented as a 0-1 integer linear programme. The basic formulation is essentially the same as that of the capacitated facilities location problem described in the section 2.1 of Chapter II. The present problem is obtained by appending the constraint that the total amount of a product required at a sales depot must be shipped from a single base depot to the capacitated facilities location problem. The reasons for appending this constraint are also described in the section 2.1 of the previous chapter. Because of this constraint, the continuous variables of the capacitated facilities location problem are now restricted to take the values zero or one only. This restriction adds to the computational complexity of the problem, obviously.

The problem being NP complete, it is very hard to solve. As a solution methodology, the Lagrangian relaxation based branch and bound procedure is proposed. A generalized assignment problem (GAP) is obtained at an intermediate stage, and is again an NP complete problem. There exist several solution methodologies for GAP. Some of these are briefly discussed in this chapter. The selection of proper solution methodology for GAP depends upon the size of the problem and the cost of the computational effort. The Balas's additive algorithm is arbitrarily selected for solving GAP.

Finally, a numerical example is presented. The sensitivity analysis, which will be used in Chapter IV while discussing the related aspects of the discrete facilities location and allocation problem, is also carried out. The steps of solution methodology for this example are given in the Appendix.

The next two sections deal with the formulation of the problem. When each product is supplied from a single source, the formulation is simple, and is presented in the section 3.1. If a product is manufactured at multiple plants, the formulation becomes complex in terms of the number of decision variables, and is presented in the section 3.2

3.1 PROBLEM FORMULATION (A PRODUCT SUPPLIED FROM A SINGLE SOURCE)

The discrete facilities location and allocation problem (P1) when a product is supplied from a single source, i.e., multiple plants with each plant manufacturing a separate set of products, can be stated as a 0-1 integer linear programme. It has been further assumed that cost parameters are linear in the quantity of the products handled.

3.1.1 Notations

The following notations are used for the formulation of the problem (P1).

Subscripts:

    i   = sales depot

    j   = potential location for a base depot

k =   product

l =   plant

k_l^1 =   group of products which are manufactured only at the plant l.

Parameters:

m =   total number of sales depots

n =   total number of base depots

p =   total number of products

q =   total number of plants ($q \leq p$)

          =   (total number of groups of products manufactured at a single plant)

C_{ijk} =   average annual cost of supplying product k to sales depot i via base depot j

F_j =   fixed cost of locating a base depot at site j

D_{ik} =   average annual demand of product k at sales depot i

Further, the following sets are also defined:

I =   set of sales depots = $\{i \mid i = 1, \ldots, m\}$

J =   set of potential base depot locations
          = $\{j \mid j = 1, \ldots, n\}$

K =   set of products = $\{k \mid k = 1, \ldots, p\}$

L =   set of plants = $\{l \mid l = 1, \ldots, q\}$

Intermediate Decision Variables:

LB_j =   lower bound on the capacity of base depot j

UB_j =   upper bound on the capacity of base depot j

N =   the number of base depots desired

Decision Variables:

$$X_{ijk} = \begin{cases} 1, & \text{if product k is supplied to sales depot i} \\ & \text{via base depot j,} \\ 0, & \text{otherwise.} \end{cases}$$

$$Y_{j} = \begin{cases} 1, & \text{if a base depot is located at site j,} \\ 0, & \text{otherwise.} \end{cases}$$

3.1.2 Formulation of (P1)

The formulation of the problem (P1) is as follows:

(P1) : **Objective Function:**

$$\text{Minimize} \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} C_{ijk} X_{ijk} + \sum_{j \in J} F_{j} Y_{j} \qquad (3.1)$$

Constraints:

(1) *No Shortages at Sales Depots* : The demand for all the products must be satisfied at all the sales depots, i.e.,

$$\sum_{j \in J} X_{ijk} = 1, \qquad i \in I, \ k \in K. \qquad (3.2)$$

(2) *Limit on the Number of Base Depots* : The number of base depots must be equal to the number of base depots desired (N), which is an intermediate decision variable, i.e.,

$$\sum_{j \in J} Y_{j} = N. \qquad (3.3)$$

(3) *Limit on the Capacity of Base Depots* : A base depot can handle the quantity within its capacity limits which are also intermediate variables, i.e.,

$$LB_{j} Y_{j} \leq \sum_{i \in I} \sum_{k \in K} D_{ik} X_{ijk} \leq UB_{j} Y_{j}, \qquad j \in J. \qquad (3.4)$$

(4) *Satisfying the Demand at a Sales Depot through a Base Depot* : A sales depot can be served from the site j provided a

base depot is located there, i.e.,

$$X_{ijk} \leq Y_j, \qquad i \in I, \; j \in J, \; k \in K. \qquad (3.5)$$

(5) *Integrality Constraints* :

$$X_{ijk} = 0 \text{ or } 1, \qquad i \in I, \; j \in J, \; k \in K,$$

$$Y_j = 0 \text{ or } 1, \qquad j \in J. \qquad (3.6)$$

It is possible to obtain a more compact integer linear programme $(P1^*)$ of $(P1)$ by replacing the constraints (3.5) with $\sum_{i \in I} X_{ijk} \leq m \, Y_j$, $j \in J$, $k \in K$. This reduces the total number of constraints from (mnp) to (np). As will be seen later, $(P1^*)$ is of no use for the proposed solution methodology.

The cost parameter C_{ijk} for product k includes freight charges from plant to sales depot i via base depot j, insurance charges, enroute entering taxes and clearing and forwarding agent's charges at the base depot j.

It may be pointed out that the number of the decision variables X_{ijk} for problem $(P1)$ can be reduced considerably without affecting the optimality. The assumption that a product is not allowed to be manufactured at more than one plant, but a plant can manufacture more than one product leads to the redefinition of the decision variable X_{ijk} as $X_{ijk_l^1}$, where

$$X_{ijk_l^1} = \begin{cases} 1, & \text{if product group } k_l^1 \text{ is supplied to sales depot} \\ & \quad \text{i via base depot j,} \\ 0, & \text{otherwise.} \end{cases}$$

This reduces the number of such decision variables from (mnp) to (mnq). It may be recalled that $(q \leq p)$.

3.2 PROBLEM FORMULATION (A PRODUCT SUPPLIED FROM MULTIPLE SOURCES)

The formulation of (P1) can be generalized by dropping the assumption that each product is supplied from a single source. The resulting more generalized problem (P2) is discussed in the present section.

3.2.1 Notations

Most of the notations defined in the section 3.1.1 are are also valid for the present problem denoted as (P2), and some are redefined below.

Parameters:

C_{ijkl} = average annual cost of supplying product k to sales depot i from plant site l via base depot j

K^r = {group of products which are manufactured only at plant l, i.e., K_l^1, $\forall$ l} $\cap$ {products which are manufactured at more than one plant}

Decision Variables:

$$X_{ijkl} = \begin{cases} 1, & \text{if product k is supplied to sales depot i} \\ & \text{via base depot j from plant site l,} \\ 0, & \text{otherwise.} \end{cases}$$

3.2.2 Formulation of (P2)

The formulation of the problem (P2) is presented below.

(P2) : Objective Function:

$$\text{Minimize} \quad \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \sum_{l \in L} C_{ijkl} X_{ijkl} + \sum_{j \in J} F_j Y_j \qquad (3.7)$$

Constraints:

The constraints for problem (P2) after modifying the respective constraints of problem (P1) are:

(1) $\sum_{j \in J} X_{ijkl} = 1,$ $i \in I, k \in K, l \in L,$ (3.8)

(2) $\sum_{j \in J} Y_j = N,$ (3.9)

(3) $LB_j Y_j \leq \sum_{i \in I} \sum_{k \in K} \sum_{l \in L} D_{ik} X_{ijkl} \leq UB_j Y_j,$ $j \in J,$ (3.10)

(4) $X_{ijkl} \leq Y_j,$ $i \in I, j \in J, k \in K, l \in L,$ (3.11)

(5) $X_{ijkl} = 0$ or $1,$ $i \in I, j \in J, k \in K, l \in L,$

     and $Y_j = 0$ or $1,$ $j \in J.$ (3.12)

It may be pointed out that the number of the decision variables X_{ijkl} for problem (P2) can also be reduced by redefining X_{ijkl} as $X_{ijk^r l}$ which takes the value 1, if product/product-group k^r is supplied to sales depot i via base depot j; 0, otherwise.

3.3 SOLUTION METHODOLOGY

The Lagrangian relaxation technique has been applied to many hard problems during the previous decades. The technique has been found very efficient, computationally, for problems which consist of a combination of hard and easy constraints. The hard constraints are dualized to obtain a Lagrangian problem which is easy to solve. The objective value of this Lagrangian problem is the lower bound on the objective value of the original problem. The lower bound obtained is at least as good as the one obtained by linear programming relaxation. Therefore, the

Lagrangian problem can be used to obtain lower bounds for branch and bound procedure instead of the linear programming relaxation. The above facts are proved analytically and more details can be found in references Parker and Rardin [13], Fisher [6] and Minoux [12].

For problem (P1), the proposed Lagrangian relaxation dualizes the constraints (3.2) and (3.4). For simplicity it is assumed that the upper bound on the capacity of the base depots is infinite. When the capacity of these base depots is finite, the number of capacity constraints, i.e., constraints (3.4) will increase from (n) to (2n). Therefore, without loss of generality the solution procedure essentially remains the same.

Some dual variables and vectors which will be used while discussing the solution methodology are defined in the next section.

3.3.1 Dual Variables and Vectors

The following dual variables and vectors are defined :

V_{ik} = multipliers for constraints (3.2), $i \in I$, $k \in K$

U_j = multipliers for constraints (3.4), $j \in J$

x = vector of X_{ijk}, $i \in I$, $j \in J$, $k \in K$

y = vector of Y_j, $j \in J$,

u = vector of U_j, $j \in J$,

v = vector of V_{ik}, $i \in I$, $k \in K$

3.3.2 Lagrangian Relaxation for Problem (P1)

The Lagrangian problem for the problem (P1) and its dual are described below.

The Lagrangian relaxation $(P1_L)$ of problem (P1) is

$$(P1_L) : \text{Minimize} \left[\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} C_{ijk} X_{ijk} + \sum_{j \in J} F_j Y_j + \sum_{j \in J} U_j \left(LB_j Y_j - \sum_{i \in I} \sum_{k \in K} D_{ik} X_{ijk} \right) + \sum_{i \in I} \sum_{k \in K} V_{ik} \left(\sum_{j \in J} X_{ijk} - 1 \right) \right]$$

s.t.   (3.3), (3.5), (3.6);

V_{ik} unrestricted, $i \in I$, $k \in K$,

$U_j \geq 0$, $j \in J$.

The problem $(P1_L)$ after rearranging the terms in the objective function can be rewritten as

$$\text{Minimize} \left[\sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) X_{ijk} + \sum_{j \in J} \left(F_j + U_j LB_j \right) Y_j - \sum_{i \in I} \sum_{k \in K} V_{ik} \right] = L(x, y, u, v)$$

s.t.   (3.3), (3.5), (3.6),

V_{ik} unrestricted, $i \in I$, $k \in K$,

$U_j \geq 0$, $j \in J$.

The corresponding Lagrangian dual, which is denoted as $(P1_D)$, is

$(P1_D) : \underset{(u,v)}{\text{Maximize}} \quad L(x,y,u,v)$

s.t. V_{ik} unrestricted, $i \in I$, $k \in K$,

$U_j \geq 0$, $j \in J$.

The Lagrangian problem $(P1_L)$ can be solved analytically without much difficulty for given u and v. From the expression for $L(x,y,u,v)$ and the constraints (3.5) of problem (P1), it is

clear that the optimal X_{ijk}'s are given by

$$X_{ijk} = \begin{cases} Y_j, & \text{if } \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) < 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$i \in I, \quad j \in J, \quad k \in K.$$

It may be noted that the compact formulation $(P1^*)$ was rejected because the set of strong constraints (3.5) helps in getting the optimal values of X_{ijk}'s while the corresponding constraints of $(P1^*)$ do not help.

Defining

$$\alpha_j(u,v) = \left[\sum_{i \in I} \sum_{k \in K} \min\left(0, \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) \right) + \left(F_j + U_j LB_j \right) \right],$$

the problem $(P1_L)$, for given u and v, can be rewritten as

$$\text{Minimize} \quad \sum_{j \in J} \alpha_j(u,v) \, Y_j - \sum_{i \in I} \sum_{k \in K} V_{ik} \tag{3.13}$$

$$\text{s.t.} \quad \sum_{j \in J} Y_j = N, \tag{3.14}$$

$$Y_j = 0 \text{ or } 1, \quad j \in J, \tag{3.15}$$

$$\text{and} \quad x_{ijk} = \begin{cases} Y_j, & \text{if } \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) < 0, \\ 0, & \text{otherwise,} \end{cases} \tag{3.16}$$

$$i \in I, \quad j \in J, \quad k \in K.$$

which is a trivial problem as by arranging $\alpha_j(u,v)$ in ascending order and picking first N of $\alpha_j(u,v)$ and assigning the value 1 to the corresponding Y_j, the problem $(P1_L)$ can be solved.

3.3.3 Solution Methodology for the Lagrangian Problem and its Dual

To solve the Lagrangian dual problem $(P1_D)$, which is a nonlinear programme, the subgradient Lagrangian search procedure proposed by Parker and Rardin [13], is used. A direction k is called an ascent direction of the function $\phi(x)$ at $\bar{x}$, if there exists some $\delta > 0$ such that $\phi(\bar{x}+\lambda k) > \phi(\bar{x})$ for all $\lambda \in (0,\delta)$. It may be pointed out that the subgradients do not always yield the ascent directions which is an inherent property of gradients. Nonetheless, the subgradients do take us closer to the optimal solution. Recognizing the weakness of subgradients, researchers have suggested the Lagrangian dual ascent procedure for solving the Lagrangian dual problem. However, applying this algorithm becomes difficult because of the difficulty in finding out ascent direction step (Parker and Rardin [13]). To find out the subgradient step size, in the present work the relaxation method discussed by Minoux [12] is used. For the choice of relaxation coefficient, various strategies are available. Held, Wolfe and Crowder [10] mention satisfactory experience with their heuristic rules for the choice of the sequence of the relaxation coefficient. Therefore, in the present work for fast convergence their heuristic rules are used.

To obtain the optimal solution to $(P1)$, the branch and bound procedure with branching variables as Y_j, $j \in J$, and solution to the dual problem $(P1_D)$ to obtain the lower bounds at the nodes of the search tree, are proposed. It may be noted that it is

possible to use branch and bound procedure with Y_j, $j \in J$, and X_{ijk}, $i \in I$, $j \in J$, $k \in K$, as branching variables and Lagrangian relaxation as the lower bounding technique. At a node of the search tree, if the constraint (3.3) is satisfied, most of the X_{ijk}, $i \in I$, $j \in J$, $k \in K$, become zero due to the constraint (3.5). Thus the remaining problem is solved using the additive algorithm for solving linear programmes with zero-one variables developed by Balas [1]. Defining $J^+ = \{j \in J \mid Y_j = 1\}$, the reduced problem $(P1_R)$ is

$(P1_R)$: Objective Function:

$$\text{Minimize} \quad \sum_{i \in I} \sum_{j \in J^+} \sum_{k \in K} C_{ijk} X_{ijk} \qquad (3.17)$$

Constraints:

$$\sum_{j \in J^+} X_{ijk} = 1, \qquad i \in I, \ k \in K, \qquad (3.18)$$

$$LB_j \leq \sum_{i \in I} \sum_{k \in K} D_{ik} X_{ijk} \leq UB_j, \qquad j \in J^+, \qquad (3.19)$$

$$X_{ijk} = 0 \text{ or } 1, \qquad i \in I, \ j \in J^+, \ k \in K. \qquad (3.20)$$

As can be seen, the problem $(P1_R)$ is the generalized assignment problem (GAP) which is again a very hard problem to solve. Some of the solution methodologies for GAP is discussed in the next section.

3.3.4 Review of Solution Methodologies for GAP

Fisher [6] has discussed two natural Lagrangian relaxations for GAP. A brief review of these relaxations is presented here.

The first relaxation suggested by Ross and Soland [15], is obtained by dualizing the constraints (3.18). In that case the GAP reduces to $(2|J^+|)$ 0-1 knapsack problems.

The second relaxation suggested by Legendre and Minoux [11], and also described in Minoux [12], is obtained by dualizing the constraints (3.19). The remaining problem is a trivial problem for given Lagrangian multipliers.

The Lagrangian dual objective function for both the relaxations is piecewise concave function. Therefore, if the subgradient Lagrangian search converges, the optimal solution to the GAP is obtained; otherwise, the branch and bound procedure is used to find out the optimal solution with Lagrangian relaxation as the lower bounding technique.

To solve the problem $(P1_R)$, therefore, many choices are available, e.g, either to solve one of the two Lagrangian relaxation problems along with the branch and bound procedure or to use the Balas's additive algorithm. The choice of the proper way depends upon the size of the problem and the cost of the computational effort. The Balas's additive algorithm is arbitrarily selected for the present work. It may be further remarked that if the constraint (3.19) is inactive (the obvious case is when the upper bound is infinitely large and the lower bound is zero) the problem $(P1_R)$ is a trivial problem.

3.3.5 Stepwise Solution Procedure

The Balas's additive procedure as outlined in Deo [3], is used. This procedure is written using the programming language PASCAL and takes the input as the coefficients in the objective function and the constraints. The output for feasible problem is the objective value and the values taken by variables. If the problem is infeasible a boolean with the value false is returned. In the PASCAL implementation of the algorithm given below, breadth first search for exploring the binary tree, is used. The step wise solution procedure is presented below.

ALGORITHM *DISCRETE FACILITIES LOCATION-ALLOCATION.*

(**PROCEDURE** *SUBGRADIENT LAGRANGIAN SEARCH* starts here).

STEP 0 : *INITIALIZATION.* Pick any $u^1 \geq 0$, v^1 unrestricted.
Set $c \leftarrow 1$, step $\leftarrow 2(m.p + n) + 1$, $\rho \leftarrow 2$, $\nu^o \leftarrow -\infty$.

STEP 1 : *LAGRANGIAN RELAXATION.* Calculate $\alpha_j(u^c, v^c)$,
$j \in J$. Solve the Lagrangian relaxation problem $(P1_L)$
using the following procedure :

IF this is the first call THEN $s \leftarrow 1$ and $Q \leftarrow J$;

OTHERWISE $Q \leftarrow T$.

IF $(s \neq 1)$ THEN set $s \leftarrow s + 1$.

WHILE $(s \leq N)$ DO

BEGIN

Find $\bar{\alpha}_{jo} = \underset{j \in Q}{\text{minimum}} \left\{ \alpha_j(u^c, v^c) \right\}$,

Assign the value 1 to the corresponding Y_j,

$$\text{set } Q \leftarrow Q - jo, \quad s \leftarrow s + 1.$$

END;

$$\text{Set } X_{ijk} \leftarrow \left\{ \begin{array}{l} Y_j, \text{ if } \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) < 0, \\ 0, \text{ otherwise,} \end{array} \right.$$

$$i \in I, \quad j \in J, \quad k \in K.$$

$$\text{IF } \left[\sum_{j \in J} U_j \left(LB_j Y_j - \sum_{i \in I} \sum_{k \in K} D_{ik} X_{ijk} \right) = 0 \right] \text{ and}$$

$$\left[\left(LB_j Y_j - \sum_{i \in I} \sum_{k \in K} D_{ik} X_{ijk} \right) \leq 0, \quad j \in J \right] \text{ and}$$

$$\left[\left(\sum_{j \in J} X_{ijk} - 1 \right) = 0, \quad i \in I, \quad k \in K \right]$$

THEN

BEGIN

IF this is the first call THEN goto STEP 5;

OTHERWISE return back to STEP 10 by

setting $\beta \leftarrow \nu^c$;

x^c and y^c give the global optimum solution or a

very good feasible local optimum solution.

END;

OTHERWISE goto STEP 2.

STEP 2 : *INCUMBENT SAVINGS.* IF $\nu^{c-1} <$ the objective value of

$(P1_L)^c$ THEN save a new dual incumbent solution by

$\nu^c \leftarrow$ the objective value of $(P1_L)^c$;

OTHERWISE set $\nu^c \leftarrow \nu^{c-1}$.

STEP 3 : *SUBGRADIENT STEP.* Compute new vectors :

$$u^{c+1} \leftarrow u^c + \lambda_c (x^c - 1) \text{ and}$$

$$v^{c+1} \leftarrow v^c + \lambda_c (LB \; y^c - D \; x^c), \text{ where}$$

$$\lambda_c = \rho_c \left(\left[L^* - L(u^c, v^c, x^c, y^c) \right] \cdot \left[||(x^c - 1)||^2 \right. \right.$$
$$\left. \left. + ||(LB \; y^c - D \; x^c)||^2 \right) \right]^{-1} \right) \text{ and}$$

L^* is the upper bound on the objective value of (P1).

IF (c = step) THEN

BEGIN

    IF $\lambda_c > \epsilon$ (fixed in advance) THEN set $\rho \leftarrow \rho / 2$.

    step $\leftarrow$ truncate (step / 2) + 1.

    IF (step < 5) THEN set step $\leftarrow$ 5.

    step $\leftarrow$ step + c.

END;

STEP 4 : *PROJECTION.* Project the new v^{c+1} on $\langle v \geq 0 \rangle$ by

setting $\quad V_{ik}^c \leftarrow \max \left\{ 0, \; V_{ik}^c \right\}, \qquad i \in I, \; k \in K.$

Set $c \leftarrow c + 1$.

IF $c = \mu$ (fixed in advance) and this is not the first

call THEN return back to STEP 10 by setting $\beta \leftarrow \nu^c$;

OTHERWISE goto STEP 5.

Otherwise goto STEP 1.

(PROCEDURE *SOLVE THE PROBLEM* ($P1_R$) starts here).

STEP 5 : IF $LB_j = 0$, $j \in J$, THEN goto STEP 6 ,($P1_R$) is trivial;

OTHERWISE goto STEP 7.

STEP 6 :   Set $P \leftarrow I$, $R \leftarrow K$.

WHILE $(P \neq \phi)$ DO

BEGIN

Select a member ι from P,

Set $P \leftarrow P - \iota$,

WHILE $(R \neq \phi)$ DO

BEGIN

Select a member k from R,

Set $R \leftarrow R - k$,

Find $C_{\iota j o k} = \underset{j \in J^+}{\text{minimum}} \left\{ C_{\iota j k} \right\}$,

IF unsuccessful THEN exist $\leftarrow$ false, come

out of the loops;

OTHERWISE, assign the value 1 to $X_{\iota j o k}$,

Set exist $\leftarrow$ true,

END.

Set $R \leftarrow K$.

END.

IF this is not the first call THEN return back to

STEP 11; OTHERWISE goto STEP 8 with $\beta \leftarrow$ objective

obtained for $(P1)$ at this stage.

STEP 7 :   *BALAS'S ADDITIVE ALGORITHM.* Solve the problem $(P1_R)$.

IF $(P1_R)$ is feasible THEN set exist $\leftarrow$ true; OTHERWISE

set exist $\leftarrow$ false. Set $\beta \leftarrow$ objective obtained for

$(P1)$ at this stage.

IF this is not the first call THEN

return back to STEP 11; OTHERWISE goto STEP 8.

(PROCEDURE *BRANCH AND BOUND* starts here.)

STEP 8 : *INITIALIZATION.* Set $\nu^* \leftarrow \beta$, where ν^* is the objective value of the problem (P1), $T \leftarrow J$. Establish the candidate list by making the full problem (P1) as its only candidate.

STEP 9 : *CANDIDATE SELECTION.* Generate a candidate $P(S_K)$ by setting Y_{jo} as zero or one, $jo \in T$, set $T \leftarrow T - jo$. Set $s \leftarrow \sum_{j \in J^-} Y_j$, where $J^- = \{j \in J \mid Y_j$ is restricted member of the candidate $P(S_K)\}$. Set $T \leftarrow T - J^-$.

STEP 10 : *BOUNDING.* IF $P(S_K)$ is not the same candidate as the first solution THEN

BEGIN

    goto STEP 1 and compute the lower bound β.

    IF $\beta \geq \nu^*$ THEN goto STEP 14;

    OTHERWISE goto STEP 11.

END;

OTHERWISE goto STEP 14.

STEP 11 : *FEASIBLE SOLUTION.*

    CASE $\sum_{j \in J} Y_j$ OF

        $> N$: the candidate is infeasible, goto STEP 14.

        $= N$: goto STEP 5 and calculate $\nu(P(S_K))$.

        $< N$: goto STEP 13.

    ENDCASE;

CASE exist OF

True : IF $\nu(P(S_K)) < \nu^*$ THEN goto STEP 12;

OTHERWISE goto STEP 13.

False : goto STEP 14.

ENDCASE;

STEP 12 : *INCUMBENT SAVING.* Save the new solution and set $\nu^* \leftarrow \nu(P(S_K))$. IF ν^* is now $-\infty$ THEN STOP; $P(1)$ is unbounded; OTHERWISE, goto STEP 14.

STEP 13 : *BRANCHING.* Replace $P(S_K)$ in the candidate list by more restricted candidate problems $P(S_{K1})$, $P(S_{K2})$, . . .,$P(S_{Kp})$. Goto STEP 9.

STEP 14 : *FATHOMING & PRUNING.* Delete $P(S_K)$ from the candidate list. IF the candidates remain THEN goto STEP 9; OTHERWISE STOP.

The above algorithm is described keeping in mind the problem (P1). Nevertheless, the same algorithm is also applicable to the problem (P2). The problem (P2) differs from the problem (P1) only in the number of dimensions involved in the decision variables. Moreover, the increase in the number of dimensions does not change the properties of Lagrangian relaxation.

3.4 A NUMERICAL EXAMPLE

The algorithm DISCRETE FACILITIES LOCATION-ALLOCATION is illustrated by the following numerical example consisting of five sales depots, five potential base depot location sites and two

product groups. The cost data are given in table 3.1, demand data in table 3.2 and fixed cost data in table 3.3. The upper limit on the capacity of base depots is taken as unlimited. Initially, the lower limit on the capacity of base depots is considered as zero. The corresponding product supply matrix which refers to the decision variables X_{ijk} is given in table 3.4. The entry 1 represents that the product k is supplied to the sales depot i via the base depot j. The other results are also given here. The details are shown in the Appendix.

Finally, the results of sensitivity analysis are presented in table 3.5 which shows the influence of the number of base depots and the lower limit on the capacity of base depots on the total cost.

3.4.1 Problem Parameters

Number of potential base depot locations = 5

Number of saled depots = 5

Number of products = 2

TABLE 3.1 : COST MATRIX (C_{ijk})

SALES DEPOT ↓	POTENTIAL BASE DEPOTS									
	1		2		3		4		5	
	PRODUCT		PRODUCT		PRODUCT		PRODUCT		PRODUCT	
	1	2	1	2	1	2	1	2	1	2
1	22	40	32	50	35	53	44	62	42	60
2	23	49	13	39	28	54	26	52	30	56
3	34	40	36	42	21	27	42	48	33	39
4	22	47	13	38	21	46	0	25	12	37
5	32	36	29	33	24	28	24	28	12	16

TABLE 3.2 : AVERAGE ANNUAL DEMAND MATRIX (D_{ik})

S. D. → PRODт	1	2	3	4	5
1	7	6	7	6	4
2	5	4	5	4	5

TABLE 3.3 : FIXED COST OF LOCATING A BASE DEPOT (F_j)

B. D. →	1	2	3	4	5
COST	50	50	48	45	45

Other Input Data :

Upper bound on the objective value = 700.

Number of base depots (N) = 2.

Lower bound on the capacity of base depots = 0.

Upper bound on the capacity of base depots (L^*) = +∞.

3.4.2 Solution

The solution to the above problem is given below. The steps of this solution are presented in the the Appendix.

The base depots' location sites are $(j = 4, 5)$.

TABLE 3.4 : PRODUCT SUPPLY ROUTE MATRIX (X_{ijk})

SALES DEPOT ↓	BASE DEPOTS			
	4		5	
	PRODUCT		PRODUCT	
	1	2	1	2
1	1	0	0	1
2	1	0	0	1
3	1	0	0	1
4	1	1	0	0
5	1	0	0	1

Total cost = 296.

3.4.3 Sensitivity Analysis

The next table shows the variation in total cost with respect to the number of base depots and the lower bound on the capacity of these base depots.

TABLE 3.5

$N \downarrow$	$L^* \rightarrow$	0,2	6	10	14
1	COST	306	306	306	306
	B.D. SITES	4	4	4	4
2	COST	296	299	306	336
	B.D. SITES	4,5	4,5	4,5	4,5
3	COST	343	349	363	363
	B.D. SITES	3,4,5	1,4,5	3,4,5	3.4.5
4	COST	393	399	423	INFEASIBLE
	B.D. SITES	1,3,4,5	1,2,4,5	2,3,4,5	——
5	COST	443	443	447	INFEASIBLE
	B.D. SITES	1,2,3,4,5	1,2,3,4,5	1,2,3,4,5	——

It can be observed from the above table that as the number of base depots established increases, the total cost also fluctuates. Here, the total cost in some cases at first decrease, then increases while for others it increases. To this cost, the cost of total system inventory should be added in order to take the final decision about the number of base depots. The procedure for calculating this total system inventory cost is discussed in the next chapter.

It is also clear that the cost increases as the lower limit on the capacity of base depots increases, but sometimes one can not avoid this high cost. The reason is that such capacity constraints are put by clearing and forwarding agents in order to safeguard their total revenue.

CHAPTER IV
RELATED ISSUES AND COMPUTATIONAL EXPERIENCE

The issues related to discrete facilities location and allocation problems are dealt with first in section 4.1. As discussed in Chapter I, in a multiechelon distribution system two types of issues arise, one with long planning horizon and the other with short planning horizon. The former category of issues comes under the discrete facilities location and allocation problems. The later category of issues are discussed here and a way is suggested to integrate these issues with the discrete facilities location and allocation problems.

The procedure Discrete Facilities Location-Allocation has been coded in PASCAL progrmming language. Here some of the experiences based on this computer implementation are given with possible theoretical explanations. The CPU times taken on Supermini Computer HP 9000/850s for some problems are also reported. The second section deals with the aspects related to the computational experience.

4.1 ISSUES RELATED TO DISCRETE FACILITIES LOCATION AND ALLOCATION PROBLEM

Before going into the details, some terms which will be used subsequently are defined below.

Customer Service Level:

The Customer Service Level (CSL) over a period of time for a sales depot is the percentage of retailers' demand satisfied from on hand inventory at that sales depot during that period.

Base Depot Service Level:

The Base Depot Service Level (BDSL) over a period of time for a base depot to a sales depot is the percentage of that sales depot's demand satisfied from on hand inventory at the base depot during that period.

Sales Depot Service Level:

The Sales Depot Service Level (SDSL) for a sales depot over a period of time is the percentage of retailers' demand satisfied from on hand inventory at that sales depot during that period provided the BDSL for this sales depot is 100%.

In this section, a way is suggested to answer the questions (Q5),(Q6) and (Q7) of Chapter I such that the total delivery cost of goods to the customers' place and the total system inventory are minimized for a desired customer service level. The procedure which will be explained shortly is valid only for the type of environment discussed in Chapter I.

For a given CSL and inventory policies to be followed at sales depots and base depots, sometimes it is possible to model the situation analytically for minimizing the system inventory depending upon the demand distributions at various levels. One such situation is modeled by Rosenbaum [14]. The following environment has been considered in that paper:

Demand distribution at sales depots   — Normally Distributed

Demand distribution at base depots   — Normally Distributed

Inventory policy for sales depots   — EOQ model

Inventory policy for base depots   — Periodic Review.

For a given CSL, the combination of BDSL/SDSL$_1$, SDSL$_2$, . . ., SDSL$_\iota$ are calculated such that the total system inventory is minimized.

BDSL and SDSLs can be used for establishing depot-norms. If it is not possible to model the situation exactly then it is suggested to resort to simulation. The procedure MULTIECHELON DISTRIBUTION SYSTEM which is presented below optimizes the complete distribution system. This procedure assumes a combination of inventory policy/policies for base and sales depots and a value for the number of base depots. The total cost is calculated. The procedure is repeated with different combinations of values of intermediate decision variables. The solution with minimum total cost is the optimal solution. The stepwise solution procedure is as follows:

STEP 0 : *INITIALIZATION.* Assume some inventory policy/ policies for the base depots and sales depots.
Set $N \leftarrow 1$ (N represents the number of base depots) and $\nu \leftarrow +\infty$ (ν represents the total cost).

STEP 1 : *DISCRETE FACILITIES LOCATION-ALLOCATION.* Use the procedure DISCRETE FACILITIES LOCATION-ALLOCATION to establish base depots and the product shipment routes. Set $\alpha \leftarrow$ total cost obtained by this procedure.

STEP 2 : *MINIMIZE SYSTEM INVENTORY.* REPEAT (for all base depots and products)
BEGIN

Calculate the combination of BDSL and SDSLs to achieve the given CSL such that the total system inventory is minimized.

END.

Calculate the cost of total system inventory.

Set $\beta \leftarrow$ total cost obtained by this procedure.

STEP 3 : *INCUMBENT SAVING*. IF $\nu < (\alpha + \beta)$, THEN save this solution and set $\nu \leftarrow (\alpha + \beta)$ and $N \leftarrow (N + 1)$.

IF $N \leq$ (potential base depot locations) THEN

goto STEP 1; OTHERWISE goto STEP 4.

STEP 4 : *INVENTORY POLICY* : IF it is possible to consider some other combination of inventory policies THEN

BEGIN

Set $N \leftarrow 1$,

goto STEP 1.

END;

OTHERWISE

STOP.

This procedure can be applied to a distribution system with any number of levels embedded in it.

4.2 COMPUTATIONAL EXPERIENCE OF DISCRETE FACILITIES LOCATION-ALLOCATION PROCEDURE

Here the issues related to the convergence of the Lagrangian dual problem $(P1_D)$ and the heuristic solutions are dealt with. The procedure suggested in Chapter III gives the exact solution

for the discrete facilities location and allocation problems. But sometimes it may become prohibitive to go for exact solution because of the cost of computational effort. Based on the computational experience, some heuristic rules are suggested with possible theoretical explanations for fast convergence of the Lagrangian dual search. Using these heuristic rules gives good intermediate solutions, therefore, one can balance the cost of quality of solution obtained with the cost of computational effort.

4.2.1 Convergence

The relaxation suggested in Chapter III suffers from the drawback that in some cases it does not allow the generation of good primal integer solutions. In the same chapter, the values of the variables x_{ijk}s have been obtained in the following manner

$$x_{ijk} = \begin{cases} Y_j, & \text{if } \left[C_{ijk} - U_j D_{ik} + V_{ik} \right] < 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\iota \in I, \quad j \in J, \quad k \in K.$$

Assuming

$i_1 \in I$, $k_1 \in K$, j_1 and $j_2 \in J$ such that $Y_j = 1$, $j \in \{j_1, j_2\}$ and denoting $\beta_{ijk} = \left[C_{ijk} - U_j D_{ik} + V_{ik} \right]$ for $i = i_1$, $j \in \{j_1, j_2\}$ and $k = k_1$, the β_{ijk} can be written as

$$\beta_{i_1 j_1 k_1} = \left[C_{i_1 j_1 k_1} - U_{j_1} D_{i_1 k_1} + V_{i_1 k_1} \right] \text{ and}$$

$$\beta_{i_1 j_2 k_1} = \left[C_{i_1 j_2 k_1} - U_{j_2} D_{i_1 k_1} + V_{i_1 k_1} \right] \text{ and}$$

$$\Delta\beta_{ijk} = \left| \beta_{i_1 j_1 k_1} - \beta_{i_1 j_2 k_1} \right|.$$

Therefore, the value of $\Delta\beta_{ijk}$ is influenced by C_{ijk} and one type of dual variable only i.e., U_j. And if the two values

$$\left(C_{i_1 j_1 k_1} - U_{j_1} D_{i_1 k_1} \right) \quad \text{and} \quad \left(C_{i_1 j_2 k_1} - U_{j_2} D_{i_1 k_1} \right) \quad \text{are}$$

approximately equal then both $X_{i_1 j_1 k_1}$ and $X_{i_1 j_2 k_1}$ will take the value 1. Hence the constraint (3.2) of the problem (P1) will not be satisfied resulting into the delay in convergence. The duality gap for some problem instances after a fixed number of iterations are given below.

Instance 1 :   (m = 5, n = 5, p = 2)

   Lower bound on the capacity of base depots = 10

   Upper bound on the capacity of base depots = +∞

   Upper bound on the optimal solution = 900

TABLE 4.1a : Quality of Bounds

N ↓	BOUND OBTAINED	OPTIMAL VALUE	$\frac{BOUND}{OPT-VAL} *100$
3	343	363	92.90
4	392.99	423	94.49

Instance 2 :   (m = 11, n = 11, p = 2)

   Lower bound on the capacity of base depots = 10

   Upper bound on the capacity of base depots = +∞

   Upper bound on the optimal solution = 2200

TABLE 4.1b : Quality of Bounds

N ↓	BOUND OBTAINED	OPTIMAL VALUE	$\frac{BOUND}{OPT-VAL} *100$
5	675.23	696	97.01

Instance 3 :   (m = 11, n = 11, p = 3)

Lower bound on the capacity of base depots = 10

Upper bound on the capacity of base depots = +∞

Upper bound on the optimal solution = 2100

TABLE 4.1c : Quality of Bounds

N ↓	BOUND OBTAINED	OPTIMAL VALUE	$\frac{\text{BOUND}}{\text{OPT-VAL}} * 100$
5	647.30	693	93.40

The next issue is related to the subgradient step. As mentioned in the previous chapter, the step size is calculated by relaxation method along with the heuristic rules of Held, Wolfe and Crowder [10]. This requires the upper bound on the optimal solution as an input parameter. This upper bound can be obtained by applying some heuristic. It has been found that for the discrete facilities location and allocation problem, if the upper bound specified is about 2-3 times the current objective value which is obtained before starting the branch and bound procedure the convergence is faster. The figure 4.1 shows this observation for some of the problem instances.

It has also been found that the initial choice of dual variables has significant influence on the duality gap after some fixed number of iterations of the procedure subgradient Lagrangian search. The following choices have been tested:
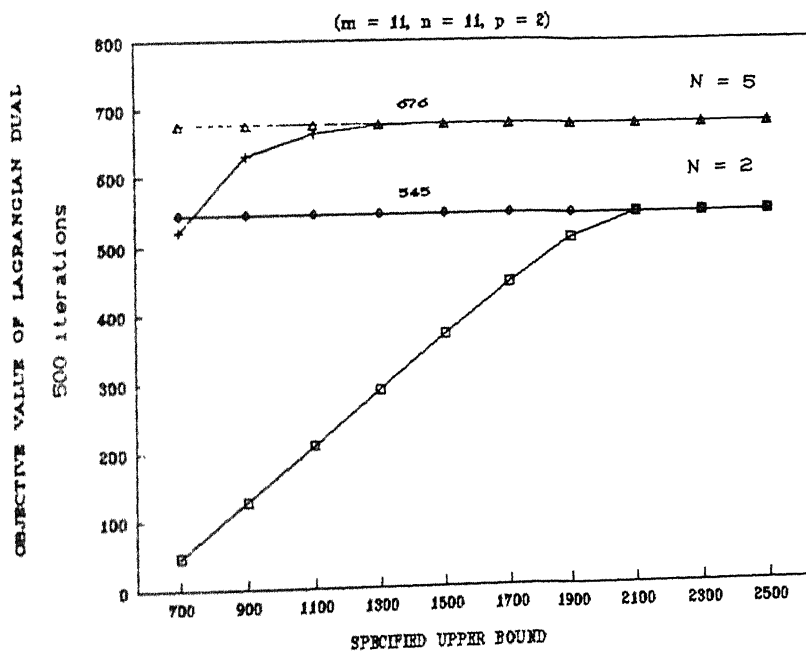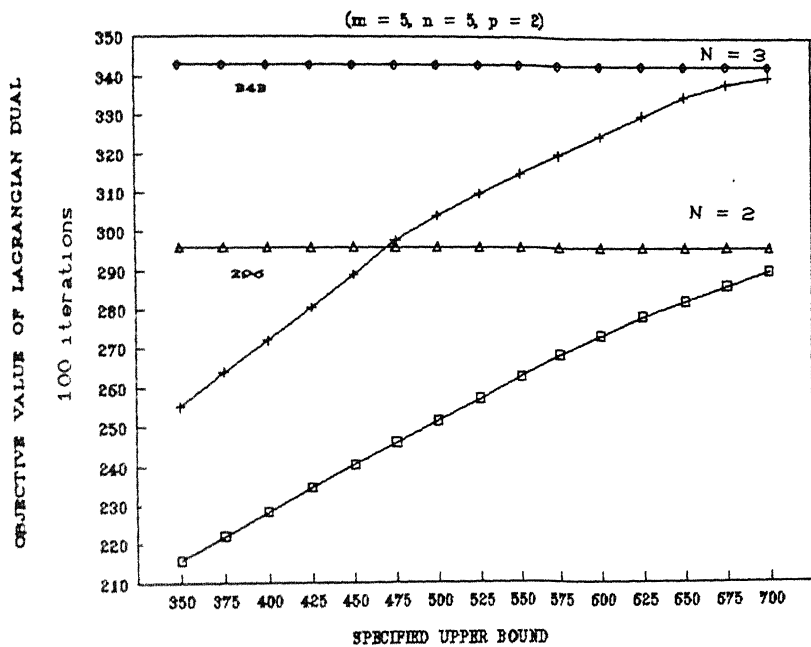
(1) U_j = $\max_{i,k} C_{ijk}$, $j \in J$ and

Figure 4.1 : Effect of Upper Bound on Convergence

$$V_{ik} = -\max_{j} C_{ijk}, \qquad i \in I, \; k \in K.$$

$$(2) \qquad U_{j} = 0, \qquad j \in J \text{ and}$$

$$V_{ik} = 0, \qquad i \in I, \; k \in K.$$

The choice (1) has been found to be superior to the choice (2). The reason lies with the economic interpretation of these dual variables, but it is definite that the first choice starts from a point closer to the optimal point in the sense of Euclidean distance.

4.2.2 Heuristic Solutions

The ways to obtain good heuristic solutions have already been explained in the previous chapter. Before starting the branch and bound procedure, the algorithm Discrete Facilities Location-Allocation obtains an upper bound on the solution. Most of the times this solution is the global optimum solution or a very good local optimum solution. At the end of this section the CPU timings needed for obtaining the first feasible solution and the global optimum solution are given. The table also gives the percentage difference between the two solutions.

CPU TIME (ON HP 9000/850s, including input/output time):

Instance 1 : (m = 5, n = 5, p = 2)

N	LB	INL. BOUND	CPU TIME sec.	OPT_VAL	CPU TIME sec.
1	0	306	0.07	306	0.07
2	0	296	1.08	296	2.49
	10	306	1.48	306	8.99
3	0	343	1.47	343	3.79
	10	363	1.58	363	18.99

Instance 2 : (m = 11, n = 11, p = 2)

LB = 0

N = 1

Initial Bound = 698,   CPU time = 0.38 seconds

Optimum value = 698,   CPU time = 0.38 seconds

CHAPTER V
CONCLUSIONS

In the present work, it has been attempted to present a solution methodology for the discrete facilities location and allocation problems faced in a consumer product company. Nevertheless, the formulation and the solution methodology are applicable to other areas also. The problem has been formulated as a pure 0-1 integer linear programme. The suggested solution methodology is to use branch and bound procedure with location variables as branching variables and Lagrangian relaxation as the lower bounding technique. The generalized assignment problem which is obtained at an intermediate stage has been solved using Balas's additive algorithm, but any available technique can be applied to solve GAP depending upon the cost of computational effort which is the characteristic of a real life problem instance.

Keeping in mind the computational complexity of the problem it is suggested to group the products into some representative groups in order to reduce the number of decision variables . Moreover, before starting the branch and bound procedure, usually a very good feasible solution is always there. It has been found that this solution is frequently the optimal solution.

The relaxation is weak, therefore it maintains significant duality gap. To reduce this gap, it is suggested to start the subgradient step from a good point. Moreover, while starting the branch and bound proceudre the upper bound specified on the

objective value which is an input parameter should be 2-3 times the current objective value.

Computer programs have been written for the solution methodology and the results are reported.

Finally, the problem has been viewed in a wider sense, i.e., as a subproblem of the multiechelon distribution system. An algorithm is also suggested for optimizing the complete distribution system, though this algorithm could not be tested because of the limited scope of this work.

Following areas may attract the attention for future research:

(1) Using some improved rule for the selection of branching variables.

(2) Using other efficient methods for solving the intermediate Generalized Assignment problem.

(3) Developing a decision support system for the design of a multiechelon distribution system integrating this problem.

It is hoped that the present work would pave the path for the future research in a wider perspective of issues which arise in a multiechelon distribution of a consumer product company.

REFERENCES

(1)   Balas, E., "An Additive Algorithm for Solving Linear Program with Zero-One variables", Operations Research, 13, 1965, pp. 517-545.

(2)   Cornuejols, G., Fisher, M. L., and Nemhauser, G. L., "Location of Bank Accounts to Optimize Float : An Analytic Study of Exact and Approximate Algorithms", Management Science, 23, 1977, pp. 789-810.

(3)   Deo, N., Kowalik, J. S., and Syslo, M. M., Discrete Optimization Algorithms with Pascal Programs, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1983.

(4)   Efroymsom, M. A., and Ray, T. L., "A Branch-Bound Algorithm for Plant Location", Operations Research, 14, 1966, pp. 361-368.

(5)   Erlenkotter, D., "A Dual-Based Procedure for Uncapacitated Facility Location", Operations Research, 26, 1978, pp. 992-1009.

(6)   Fisher, M. L., "The Lagrangian Relaxation Method for Solving Integer Programming Problems", Management Science, 27, 1981, pp. 1-18.

(7)   Francis, R. L., and White, J. A., Facility Layout and Location : An Analytical Approach, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1974).

(8)   Geoffrion, A. M., "Lagrangian Relaxation for Integer
       Programming", Mathematical Programming Study, 2, 1974, pp.
       82-114.

(9)   Geoffrion , A. M., and McBride, R., "Lagrangian Relaxation
       Applied to Capacitated Facility Location Problems", AIIE
       Trans., 10, 1978, pp. 40-47.

(10)   Held, M., Wolfe, P., and Crowder, H. P., "Validation of
       Subgradient Optimization", Mathematical Programming, 6,
       1974, pp. 62-88.

(11)   Legendre, J. P., and Minoux, M., "Une Application de la
       Notion de dualité en Programmation en Nombres Entries :
       sélection et Affectation Optimals D'une Flotte D'avions",
       RAIRO, 11, pp. 201-222.

(12)   Minoux, M., (translated by Vajda, S.), Mathematical
       Progrmming : Theory and Algorithms, John Wiley & Sons ,
       New York, 1986.

(13)   Parker, R. G., and Rardin, R. L., Discrete Optimization,
       Academic Press, San Diego, CA, 1988.

(14)   Rosenbaum, B. A., "Service Level Relationships in a
       Multiechelon Inventory System", Management Science, 27,
       1981, pp. 926-945.

(15)   Ross, G. T., and Soland, R. M., "A Branch and Bound
       Algorithm for the Generalized Assignment Problem",
       Mathematical Programming, 8, 1975, pp. 91-103.

(16)   Sharma, R. R. K., Multistage Bulk Commodity Distribution :
        Development of Decomposition-Lagrangian Relaxation Based
        Algorithm,   Unpublished   Fellowship   in   Management
        Dissertation, Indian Institute of Management, Ahmedabad,
        March 1988.

(17)   Van Roy, T. J., "A Cross Decomposition Algorithm for
        Capacitated Facility Location", Operations Research, 34,
        1986, pp. 145-163.

APPENDIX

This Appendix gives the steps of the solution procedure of the numerical example presented in the section 3.4 of Chapter IV. The number of iterations of Lagrangian dual search is 500, therefore this portion is skipped by presenting a few details. The stepwise solution procedure is as follows.

(1) At the STEP 1, some variables are initialized. The values of u and v vectors are as follows.

V_{11} = -34	U_1 = 62
V_{21} = -36	U_2 = 56
V_{31} = -35	U_3 = 48
V_{41} = -44	U_4 = 46
V_{51} = -42	U_5 = 33
V_{11} = -49	
V_{21} = -50	
V_{31} = -54	
V_{41} = -62	
V_{51} = -60	

The value of the variable *step* is $2(mp+n)+1$, i.e., 31. Therefore, the value of ρ will be 2 for 30 steps, then both *step* and ρ will be divided by 2 till $\lambda_c \leq \varepsilon$ (a small real number). If the variable *step* < 5, then this will be set to 5.

(2) After 500 iterations ν = 295.99 , and
$$Y_1 = 0, \ Y_2 = 0, \ Y_3 = 0, \ Y_4 = 1, \ Y_5 = 1,$$
and since this is the first call, therefore, the next step is STEP 5.

(3) Since lower bound on capacity for all the base depots is zero, the procedure invokes STEP 6. The STEP 6 solves the generalized assignment problem, because the values of location variables are taken as in step (2) above. The output is
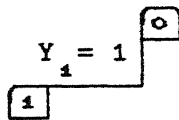
Total cost = 296.

$$X_{151} = X_{251} = X_{351} = X_{451} = X_{551}$$

$$= X_{152} = X_{252} = X_{352} = X_{452} = X_{552} = 1 \text{ and other } X_{ijk} \text{ are } 0.$$

(4) Now the procedure branch and bound starts. The STEP 8 initializes ν^* as 296 (the objective value obtained in either STEP 6 or STEP 7).
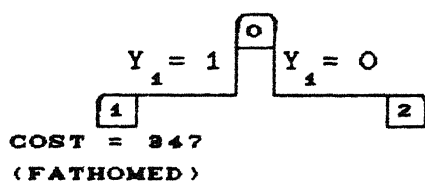
(5) A candidate is selected with Y_j (arbitrarily chosen) as branching variables. The current search tree is given below.

(6) Since, this is not the same candidate as the first solution, the procedure Lagrangian relaxation is invoked. The bound obtained for the candidate 1 is $\beta = 347$. The value of β (347) is greater than ν^* (296), therefore the candidate 1 is fathomed (STEP 14).

COST = 347
(FATHOMED)
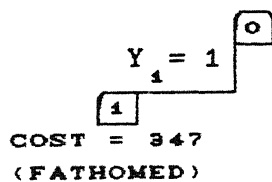
(7)   Now the next candidate is generated:

```
              ┌───┐
       Y  = 1 │ 0 │ Y  = 0
        1     └───┘  1
       ┌─┐            ┌─┐
       │1│            │2│
       └─┘            └─┘
  COST = 347
  (FATHOMED)
```
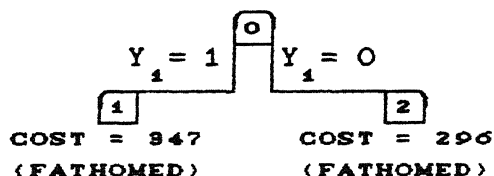
(8)   Since, this is not the same candidate as the first solution, the procedure Lagrangian relaxation is invoked. The bound obtained for the candidate 2 is $\beta = 296$. The value of β (296) is same as ν^{*} (296), therefore the candidate 1 is fathomed (STEP 14).

```
              ┌───┐
       Y  = 1 │ 0 │ Y  = 0
        1     └───┘  1
       ┌─┐            ┌─┐
       │1│            │2│
       └─┘            └─┘
  COST = 347      COST = 296
  (FATHOMED)      (FATHOMED)
```

(9)   At this stage, the candidate list is exhausted. Therefore, the procedure terminates with 296 as the optimal cost.

DISCUSSION

The questions asked and suggested ideas during the presentation of the thesis are presented here.

1. Why does the choice of starting Lagrangian multipliers (u,v) as $(0,0)$ not perform better than the choice of (u,v) as $(\max\limits_{i,k} C_{ijk}$ for $j \in J$, $V_{ik} = -\max\limits_{j} C_{ijk}$ for $i \in I$, $k \in K)$?

Answer: The optimal value of the variables X_{ijk} for a given multiplier vector (u,v) is obtained in the following manner

$$ X_{ijk} = \left[\begin{array}{l} Y_j, \text{ if } \left(C_{ijk} - U_j D_{ik} + V_{ik} \right) < 0, \\ 0, \text{ otherwise,} \end{array} \right. $$

$$ i \in I, \ j \in J, \ k \in K. $$

Hence, if the vector (u,v) is $(0,0)$ then all the X_{ijk} variables will take the value zero and the constraint (3.2) will not be satisfied resulting into the delay in convergence.

2. How can you say that the problem formulated is NP complete ?

Answer: Krarup has proved that a pure 0-1 integer linear program is NP complete. The proof is also given in the reference Parker and Rardin [13].

3.   Did you solve the problem using only Balas's algorithm ?

Answer: The Balas's additive algorithm is combinatorial in nature.   Moreover, the algorithm is valid only for one type of constraint ,i.e., $Ax \leq b$.   Therefore, for large number of variables the input matrix A is very large and sparse and does not fit into the memory of a microcomputer.   But again this depends upon the implementation of the algorithm.   From the nature of this matrix A, it is clear that this algorithm is not efficient for our problem.

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